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On obtaining classical mechanics from quantum mechanics

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Abstract

Constructing a classical mechanical system associated with a given quantum mechanical one, entails construction of a classical phase space and a corresponding Hamiltonian function from the available quantum structures and a notion of coarser observations. The Hilbert space of any quantum mechanical system naturally has the structure of an infinite dimensional symplectic manifold ('quantum phase space'). There is also a systematic, quotienting procedure which imparts a bundle structure to the quantum phase space and extracts a classical phase space as the base space. This works straight forwardly when the Hilbert space carries weakly continuous representation of the Heisenberg group and one recovers the linear classical phase space $\mathbb{R}^{2\mathrm{N}}$. We report on how the procedure also allows extraction of non-linear classical phase spaces and illustrate it for Hilbert spaces being finite dimensional (spin-j systems), infinite dimensional but separable (particle on a circle) and infinite dimensional but non-separable (Polymer quantization). To construct a corresponding classical dynamics, one needs to choose a suitable section and identify an effective Hamiltonian. The effective dynamics mirrors the quantum dynamics provided the section satisfies conditions of semiclassicality and tangentiality.

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Developing a semi-classical approximation to quantum dynamics is in general a non-trivial task. Intuitively, such an approximation entails an *adequate* class of observable quantities (eg. expectation values of self-adjoint operators) whose time evolution, dictated by quantum dynamics, is *well approximated* by *a classical Hamiltonian evolution*. Roughly, the adequate class refers to (say) basic functions on a classical phase space (symplectic manifold) with a Hamiltonian which is a function of these basic functions. The accuracy of an approximation is controlled by how well the classically evolved observables stay close to the quantum evolved ones within a given precision specified in terms of bounds on quantum uncertainties. Having a description of the quantum framework as similar as possible to a classical framework is obviously an aid in developing semi-classical approximations.

Such a description is indeed available and is referred to as geometrical formulation of quantum mechanics [1]. The quantum mechanical state space, a projective Hilbert space, is naturally a symplectic manifold, usually infinite dimensional (finite dimensional for spin systems). Furthermore, dynamics specified by a Schrodinger equation is a Hamiltonian evolution. This is true for all quantum mechanical systems. In addition, there is also a systematic quotienting procedure to construct an associated Hamiltonian system (usually of lower and mostly finite dimensions) which views the quantum state space as a bundle with the classical phase space as its base space. This works elegantly when the quantum Hilbert space is obtained as the weakly continuous representation of a Heisenberg group. Generically these are separable Hilbert spaces and the extracted classical phase spaces are *linear*, $\mathbb{R}^{2n}$.

Quantum mechanical Hilbert spaces however arise in many different ways. For example, the (kinematical) Hilbert space of loop quantum cosmology carries a *non-weakly continuous* representation of the Heisenberg group and is non-separable. For examples such as particle on a circle and spin systems, one does not even have the Heisenberg group. A semi-classical approximation is still needed for such systems. Likewise, in classical mechanics (even for finitely many degrees of freedom), the classical phase space is not necessarily *linear* (eg the cylinder for particle on a circle, reduced phase spaces of constrained systems etc). It is important to develop a quotienting procedure to construct such, possibly non-linear, classical phase spaces from more general quantum state spaces. In this work we develop such a procedure and illustrate it for three examples: arbitrary spin-J system, particle on a circle and Bohr or polymer quantization appearing in loop quantum cosmology (LQC). This takes care of the kinematical aspects.

To construct an associated classical dynamics one has also to obtain a Hamiltonian function (an *effective Hamiltonian*) on the classical phase space. This is done by choosing a section of the bundle and obtaining the effective Hamiltonian on the base space as a pull back of the quantum mechanically defined one. An effective Hamiltonian so defined, depends on the section chosen. One can now construct two trajectories on the classical phase space: (a) projection of a *quantum trajectory* (i.e. trajectory in the quantum state space) onto the base space and (b) a trajectory in the base space, generated by the effective Hamiltonian function. In general, i.e. for arbitrary sections, these two trajectories do *not* coincide. They do so when the section is tangential to the quantum trajectories (equivalently when the section is preserved by quantum dynamics). Since the classical states are obtained from expectation values (via the quotienting procedure), for the classical trajectories to reflect the quantum one, *within a certain approximation*, it is necessary that the quantum uncertainties also remain bounded within *prescribed tolerances*. In other words, the states in the section should also satisfy conditions of semiclassicality.

In section I, we recall the basic details of the geometric formulation from [1] and describe the quotienting procedure in a general setting. In general the classical phase is obtained as a *sub-manifold* of the base space of the bundle. The general procedure is then illustrated with three examples in the three subsections.

Section II contains a discussion of dynamics and the conditions for developing semi-classical approximation. The classical phase space being not the same as the base space in general, puts a first requirement that the projection of sufficiently many quantum trajectories onto the base space should actually be confined to the sub-manifold of classical phase space. For constructing a Hamiltonian dynamics on the classical phase space, a section over the classical phase space needs to be chosen which provides an embedding of the classical phase space into the quantum state space. For having a useful semiclassical approximation the section has to satisfy two main conditions of semiclassicality and tangentiality. The state space of polymer quantization brings out further points to pay attention to while developing a semi-classical approximation.

In the last section we conclude with a summary and a discussions.

I. KINEMATICAL SET-UP

This section is divided in five subsections. The first one recalls the symplectic geometry of the quantum state space. This is included for a self contained reading as well as for fixing the notation, experts may safely skip this section. More details are available in [1, 2]. The second one describes the quotienting procedure in some generality (but still restricted to finitely many classical degrees of freedom) to get candidate classical phases spaces which could also be non-linear. The next three sub-sections illustrate the quotienting procedure for the examples of spin-j system, particle on a circle and isotropic, vacuum loop quantum cosmology.

A. Symplectic geometry of (projective) Hilbert space

Let $\mathcal{H}$ be a complex Hilbert space, possibly non-separable and let $\mathcal{P}$ be the corresponding projective Hilbert space: the set of equivalence classes of non-zero vectors of the Hilbert space modulo scaling by non-zero complex numbers. Equivalently, if $\mathcal{S}$ denotes the subset of normalized vectors, then $\mathcal{P} = \mathcal{S}/$phase equivalence. Unless stated otherwise, the Hilbert space is assumed to be infinite dimensional.

Any complex vector space can be viewed as a real vector space with an almost complex structure defined by a bounded linear operator J, $J^2 = -\mathbb{I}$ defined on it and multiplication by a complex number $a+ib$ being represented as $a+bJ$. The Hermitian inner product of the complex Hilbert space can then be expressed in terms of a symmetric and an anti-symmetric quadratic forms, $G(\cdot,\cdot)$ and $\Omega(\cdot,\cdot)$ respectively. Explicitly,

$$\langle\Psi,\Phi\rangle \ := \ \tfrac{1}{2\hbar}G(\Psi,\Phi) + \tfrac{i}{2\hbar}\Omega(\Psi,\Phi). \tag{1}$$

The non-degeneracy of the Hermitian inner product then implies that G and Ω are (strongly) non-degenerate. The real part G will play no role in this paper.

For a separable Hilbert space, let $|e_n\rangle$ denote an orthonormal basis so that we have $|\Psi\rangle = \sum_n \psi^n |e_n\rangle$, $\psi^n \in \mathbb{C}$. Writing $\psi^n := x^n + iy^n$ and using the definitions of G, Ω, one can see that $\frac{1}{2\hbar}G(\Psi,\Psi') = \sum_n x^n x'^n + y^n y'^n$ and $\frac{1}{2\hbar}\Omega(\Psi,\Psi') = \sum_n x^n y'^n - y^n x'^n$. Viewing (x^n, y^n) as (global) 'coordinates' on $\mathcal{H}$ one can see that G corresponds to the infinite order identity matrix while Ω corresponds to the infinite dimensional analogue of the canonical form of a symplectic matrix. The coordinates are Darboux coordinates with

y^n, x^n as generalized coordinates and momenta, respectively. A change of basis effected by a unitary transformation just corresponds to an orthosymplectic transformation, analogue of the result that $U(n, \mathbb{C})$ is isomorphic to $OSp(2n, \mathbb{R})$. A separable Hilbert space can thus be viewed as the $N \to \infty$ form of the usual phase space $\mathbb{R}^{2N}$. A non-separable Hilbert space does not admit a countable basis and an identification as above is not possible. Much of the finite dimensional intuition from $\mathbb{R}^{2N}$ can be borrowed for separable Hilbert spaces.

The real vector space can naturally be thought of as a (infinite dimensional) manifold with tangent spaces at each point being identified with the vector space itself. Explicitly, a tangent vector $|\Phi\rangle$ at a point $\Psi \in \mathcal{H}$ acts on real valued functions, $f(\Psi) : \mathcal{H} \to \mathbb{R}$ as,

$$|\Phi\rangle|_\Psi(f) \ := \ \lim_{\epsilon \to 0} \tfrac{f(\Psi + \epsilon\Phi) - f(\Psi)}{\epsilon} \tag{2}$$

Clearly, every vector in $\mathcal{H}$ can be viewed as a *constant* vector field on $\mathcal{H}$ viewed as a manifold. The Ω introduced above is then a non-degenerate 2-form on $\mathcal{H}$ which is trivially closed and hence defines a symplectic structure on $\mathcal{H}$. This immediately allows one to define, for every once differentiable function $f(\Psi)$, a corresponding *Hamiltonian vector field*, X_f, via the equation: $\Omega(X_f, Y) = Y(f)\ \forall$ vector fields Y. Non-degeneracy of Ω implies that Hamiltonian vector fields are uniquely determined. The Poisson bracket between two such functions, f, g, is then defined as: $\{f, g\}_q := \Omega(X_f, X_g)$.

Next, for every self adjoint operator $\hat{F}$, we get a non-constant vector field $X_F|_\Psi := -\frac{i}{\hbar}\hat{F}|\Psi\rangle$ and in analogy with the Schrodinger equation, it is referred to as a Schrodinger vector field. This vector field turns out to be a *Hamiltonian vector field* i.e. there exist a function $f(\Psi)$ such that $\Omega(X_F, Y) = df(Y) := Y(f)$ for all vector fields Y. From the definition of X_F and of Ω in terms of the inner product, it follows that $\Omega(X_F, Y)|_\Psi = \langle \hat{F}\Psi, Y|_\Psi\rangle + \langle Y|_\Psi, \hat{F}\Psi\rangle$. This is exactly equal to $Y(f)$ for all vector fields Y *provided* we define $f(\psi) := \langle \Psi, \hat{F}\Psi\rangle$. Thus, every self-adjoint operator defines a Schrodinger vector field which is also Hamiltonian with respect to the symplectic structure and the corresponding 'Hamiltonian function' is the expectation value (up to the norm of Ψ) of the operator. This function is quadratic in its argument and is invariant under multiplication of Ψ by phases. We will mostly be concerned with such quadratic functions. For any two such quadratic functions, $f(\Psi) := \langle \Psi, \hat{F}\Psi\rangle, g(\Psi) := \langle \Psi, \hat{G}\Psi\rangle$, the Poisson bracket $\{f, g\}_q = \Omega(X_F, X_G)$ evaluates to $\langle \frac{1}{i\hbar}\left[\hat{F}, \hat{G}\right]\rangle$. It follows that the quantum mechanical evolution of expectation value functions is exactly given by a Hamiltonian evolution:

$$\frac{d}{dt}f(\Psi) = \frac{d}{dt}\langle\Psi, \hat{F}\Psi\rangle = \langle\Psi, (i\hbar)^{-1}\left[\hat{F}, \hat{H}\right]\Psi\rangle = \{f(\Psi), h(\Psi)\}_q \tag{3}$$

This shows how the quantum dynamics in a Hilbert space can be viewed as Hamiltonian dynamics in an infinite dimensional symplectic manifold.

Quantum mechanical state space consists of *rays* or elements of the projective Hilbert space. One can import the Hamiltonian framework to the projective Hilbert space as well [1, 2]. In the first step restrict attention to the subset $\mathcal{S}$ of normalized vectors (the quadratic functions defined above now exactly become the expectation values). The projective Hilbert space can then be viewed as equivalence classes of the relation: $\Psi \sim \Psi' \Leftrightarrow \Psi' = e^{i\alpha}\Psi$. There is a natural projection $\rho : \Psi \in \mathcal{S} \to [\Psi] \in \mathcal{P}$ and the natural inclusion $i : \mathcal{S} \to \mathcal{H}$. With the inclusion map we can pull back the symplectic form Ω to $\mathcal{S}$ on which it is *degenerate*. Furthermore, the degenerate subspace gets projected to zero under ρ_* and hence, there exist a *non-degenerate* 2-form ω on $\mathcal{P}$ such that $i^*\Omega = \rho^*\omega$. This is also closed and thus endows the projective Hilbert space with a symplectic structure [1, 2]. The quadratic functions defined on $\mathcal{H}$ are automatically phase invariant and thus project down uniquely to functions on $\mathcal{P}$. In particular, we also get Poisson brackets among the projected quadratic function satisfying $\{f, g\}_q = \omega(X_f, X_g)$. In this manner, one obtains a Hamiltonian description of quantum dynamics.

B. Quotienting Procedure

Now consider a set of self-adjoint operators $\hat{F}_i, i = 1, \cdots, n$ with the corresponding quadratic functions $f_i : \mathcal{P} \to \mathbb{R}$. Define an equivalence relation on $\mathcal{P}$ which identifies two states if they have the same expectation values, f_i, of all the $\hat{F}_i$ operators. Let Γ denote the set of equivalence classes. The values $x^i = f_i(\Psi)$ naturally label the points of Γ. We will assume that Γ can be viewed as a region in $\mathbb{R}^n$ with x^i serving as coordinates. In other words, the topology and manifold structure on Γ is assumed to be inherited from the standard ones on $\mathbb{R}^n$. The manifold structure of the quantum phase space is discussed in [2]. There is a natural projection $\pi : \mathcal{P} \to \Gamma$.

We have two natural subspaces of $T_\Psi\mathcal{P}$: (a) the *vertical* subspace $\mathcal{V}_\Psi$ consisting of vectors which project to zero i.e. $\pi_* v_\Psi = 0 \in T_{\pi(\Psi)}\Gamma$ and (b) the subspace $(\mathcal{V}_F)_\Psi$ which is the span of the Hamiltonian vector fields $\{X_{f_i}, i = 1, \cdots, n\}$. For notational simplicity, we will suppress the suffix Ψ on these subspaces. The vertical vectors naturally annihilate functions which are constant over the fibre and are thus tangential to the fibre.

Let $\mathcal{V}_F^\perp$, denote the ω-complement of $\mathcal{V}_F$ i.e. $v \in \mathcal{V}_F^\perp$ implies that $\omega(X_{f_i}, v) = v(f_i) = 0\ \forall\, i$. Since f_i are constant over a fibre, every vertical vector satisfies this condition and thus belongs to $\mathcal{V}_F^\perp$. Are there vectors in $\mathcal{V}^\perp$ which are *not* vertical? This is a little subtle. It is easy to see that a vector in $\mathcal{V}_F^\perp$ annihilates all functions polynomial in f_i. One can consider a function algebra generated by f_i with suitable restrictions on f_i (and hence on $\hat{F}_i$) and completed with some suitable norm. Then all elements of such a function algebra will also be annihilated by elements of $\mathcal{V}_F^\perp$. All these functions are of course constant over the fibre. If the class of functions annihilated by the vertical vectors coincides with the function algebra, then the subspace $\mathcal{V}_F^\perp$ coincides with the vertical subspace otherwise $\mathcal{V} \subset \mathcal{V}_F^\perp$. We will ignore such fine prints and simply assume that appropriate choices can be made so that the vertical subspace *is* the ω-complement of $\mathcal{V}_F : \mathcal{V} = \mathcal{V}_F^\perp$. Using this identification, we will refer to $\mathcal{V}_F^\perp$ as the vertical subspace. At this stage, we do *not* know if $\mathcal{V}_F$ is symplectic subspace or not. If and only if $\mathcal{V}_F$ is symplectic (i.e. ω restricted to $\mathcal{V}_F$ is non-degenerate), one has (i) $\mathcal{V}_F \cap \mathcal{V}_F^\perp = \{0\}$ *and* (ii) $T_\Psi(\mathcal{P}) = \mathcal{V}_F \oplus \mathcal{V}_F^\perp$. If, however, $\mathcal{V}_F$ is *not* symplectic, then the intersection of the symplectic complements is a non-trivial subspace *and* the *vector sum* of the two is a *proper* subspace of the tangent space[1]. In the symplectic case, it is appropriate to refer to $\mathcal{V}_F$ as the *horizontal* subspace. In the non-symplectic case, this terminology for $\mathcal{V}_F$ is in-appropriate since there are tangent vectors which are not in $\mathcal{V}_F + \mathcal{V}_F^\perp$, are also intuitively 'horizontal' or transversal to the fibre. In general, we will refer to vector fields valued in $\mathcal{V}_F$ as *basal vector fields.*

These two cases are precisely distinguished by the non-singularity and singularity of the matrix of Poisson brackets, $A_{ij} := \omega(X_{f_i}, X_{f_j}) = \{f_i, f_j\}_q$ respectively. This is because, if the matrix A_{ij} is singular, then there exist linear combinations, $Y_a := \alpha^i_a X_{f_i}, a = 1, \cdots, m$, of vectors of $\mathcal{V}_F$ which are in $\mathcal{V}_F^\perp$ (i.e. $\alpha^i_a A_{ij} = 0$ has non-trivial solutions) and thus have a non-trivial intersection. Clearly, $\mathcal{V}_F$ is not a symplectic subspace in this case.

Suppose now that in addition, $\hat{F}_i$ are closed under the commutator or equivalently f_i

[1] This follows by noting that for any non-trivial subspace, W, of a symplectic space, V, $(W + W^\perp)^\perp = W \cap W^\perp$ and $V^\perp = \{0\}$.

are closed under $\{\cdot,\cdot\}_q$. Then, independent of the (non-)singularity of A_{ij}, the following statements are true by virtue of the Poisson bracket closure property of f_i : (i) commutator of two vertical vector fields is a vertical vector field and hence the vertical subspaces define an integrable distribution on $\mathcal{P}$ (this of course does not need the closure property); (ii) the commutator of two basal vector fields is a basal vector field because of the Poisson bracket closure and hence the $\mathcal{V}_F$ subspaces also define an integrable distribution on $\mathcal{P}$; (iii) finally, the commutator of a vertical vector field and a basal vector field is a vertical vector field since $\omega(X_{f_i}, [v, X_{f_j}]) = [v, X_{f_j}](f_i) = v(\{f_j, f_i\}_q) = 0$. Thus, $\mathcal{L}_v X \in \mathcal{V}_F^{\perp},\ \forall X \in \mathcal{V}_F\ , \forall v \in \mathcal{V}_F^{\perp}$. The last equality follows from the closure property of the Poisson brackets. Due to this last property, the push-forward π_* of $X \in \mathcal{V}_F$ from any point along a fibre, gives the same (i.e. a well defined) vector field on Γ. The vectors Y_a project to the 'zero' vector field on Γ. Thus $\pi_*(\mathcal{V}_F)$ is a subspace of dimension $(n-m)$ of the tangent space of Γ.

One can make the projection π_* explicit in the following manner. Consider projection of X_{f_i}. By definition, for functions $f(x^i)$ on Γ, $[\pi_* X_{f_i}](f) = X_{f_i}(\pi^* f)$. The pull-back function is constant over the fibre and it depends on Ψ only through $f_i(\Psi)$. Hence,

$$X_{f_i}(\pi^* f) \ = \ \tfrac{\partial \pi^*(f)}{\partial f_j} X_{f_i}(f_j) \ = \ \{f_i, f_j\}_q \tfrac{\partial f}{\partial x^j} \ = \ A_{ij} \tfrac{\partial}{\partial x^j} f. \tag{4}$$

Thus, $\xi_i := [\pi_* X_{f_i}] = A_{ij} \frac{\partial}{\partial x^j}$. It follows immediately that $[\pi_* Y_a] = \alpha^i_a A_{ij} \frac{\partial}{\partial x^j} = 0$ as noted above. This also implies that only $(n-m)$ of these vectors in $T_{\vec{x}}(\Gamma)$ are independent. Noting that A_{ij} being linear combinations of the f_i's, are constant along the fibres and thus descend to Γ as the same linear combinations of x^i. Computing the commutator of the ξ_i directly and using the Jacobi identity satisfied by the structure constants, one can verify that the projected vectors fields, $\pi_* X_{f_i}$ are also closed under commutators (with the same structure constants). Thus, $\pi_*(\mathcal{V}_F)$ define an integrable distribution on Γ. The integral sub-manifolds are candidate *classical phases spaces*, Γ_{cl}, with a symplectic form α (defined below) satisfying, $\omega = \pi^* \alpha$.

Let $Z_I := \beta^i_I X_{f_i},\ I = 1, \cdots, (n-m)$ be independent linear combinations of the vector fields X_{f_i} which are closed under the commutator bracket and A_{ij} is nonsingular on their span. Their projections are given as $\zeta_I := [\pi_* Z_I] = \beta^i_I \xi_i$ and are tangential to the integral sub-manifolds. If, in addition, the ζ_I's commute, then the parameters of their integral curves provide local coordinates on these integral sub-manifolds. In some cases (see the examples discussed in the subsections), the integral sub-manifolds of Γ are defined by m

equations, $\phi_a(x^i) = $ constant, with the functions satisfying $\zeta_I(\phi_a) = 0\ \forall I$. The integral sub-manifolds then are embedded sub-manifolds. In general, however, one only gets immersed sub-manifold.

Now we would like to define a symplectic form on Γ_{cl}. This can be done in two steps. Let $s : \Gamma \to \mathcal{P}$ be a section. From this one gets the pull-back, $\tilde{\omega} := s^*\omega$ on Γ. This is a closed two form but degenerate if A_{ij} is degenerate. Since any Γ_{cl} is a sub-manifold, we also get a closed two form α on Γ_{cl} via the pull-back of the inclusion map, $\alpha := i^*\tilde{\omega} = i^* \circ s^*\omega$. Explicitly,

$$\alpha(\zeta_I, \zeta_J) = \tilde{\omega}(i_*\zeta_I, i_*\zeta_J) = \tilde{\omega}(\zeta_I, \zeta_J) = \omega(s_*\zeta_I, s_*\zeta_J)\ . \tag{5}$$

Since $\pi_* s_* \zeta_I = \zeta_I$, one sees that $s_*\zeta_I = Z_I + v_I$ for some $v_I \in \mathcal{V}_F \cap \mathcal{V}_F^{\perp}$. Clearly, $\omega(Z_I + v_I, Z_J + v_J) = \omega(Z_I, Z_J)$ and α is well defined. That $\omega(v_I, v_J) = 0$ follows because both vectors are vertical as well as basal.

The 2-form α also turns out to be independent of the section chosen. To see this, observe that Lie derivative of ω along a *vertical* vector field v, when evaluated on basal vector fields X, Y, vanishes:

$$[\mathcal{L}_v\omega](X, Y) = [d(i_v\omega)](X, Y) = X(\omega(v, Y)) - Y(\omega(v, X)) - \omega(v, \mathcal{L}_X Y) = 0\ . \tag{6}$$

Here we used the facts that the vertical and the basal spaces are symplectic complements and that the commutator of basal vectors is basal. Thus, the 2-form α is well defined, independent of section, closed since it is a pull-back of a closed form and non-degenerate because ω is non-degenerate on the subspace spanned by Z_I's. The definition of α is extended to all vector fields on the integral sub-manifolds by linearity. Thus each of the sub-manifolds is now equipped with a symplectic structure and is a candidate classical phase space which will be generically denoted as Γ_{cl}. Note that the symplectic structure on the integral sub-manifolds is *independent* of the section chosen in the intermediate steps. It however, depends on the particular Γ_{cl}, via the inclusion map.

If A_{ij} is non-singular, Γ itself is the classical phase space. Consider the usual case of $\{\hat{F}_i\} = \{\hat{Q}^a, \hat{P}_a, \hat{\mathbb{I}},\ a = 1, \cdots, m\}$ forming the Heisenberg Lie algebra. The corresponding quadratic functions are $q^a(\Psi), p_a(\Psi)$ and the constant function with value 1. The number of corresponding Hamiltonian vector fields are however one less, since $X_{`1'} = 0$. Furthermore, the space Γ is *a single* hyper-plane in $\mathbb{R}^{2m+1}$. We can thus take $\Gamma = \mathbb{R}^{2m}$ and focus only on the non-trivial functions. The matrix of Poisson brackets is then non-singular. There are no vectors which are both basal and vertical and Γ itself is the classical phase space. This case is discussed in detail in [1, 2].

In summary: In this subsection, we have seen that for every Lie algebra defined by the self adjoint operators $\hat{F}_i$, one can develop a natural quotienting procedure and construct a classical phase space Γ_{cl}. The classical phase space is in general an immersed sub-manifold of the space of equivalence classes, Γ. This procedure is capable of yielding linear as well as non-linear classical phase spaces. The linear versus non-linear cases are distinguished by the (non-)singularity of the matrix of Poisson brackets, A_{ij}. The vector fields on Γ which connect different Γ_{cl} are projections of vector fields which are valued in $T_\Psi(\mathcal{P}) - (\mathcal{V}_F + \mathcal{V}_F^\perp)$. The choice of the basic operators is naturally made if the Hilbert space itself is obtained as a representation of the corresponding Lie group (as one would do in the reverse process of quantization). If however, the Hilbert space is not so chosen, then the choice is to be made appealing to the purpose of constructing a classical phase space. Our reason for constructing a classical phase space has been to develop a semiclassical approximation and this involves dynamics as well. Thus, the choice of algebra will be dictated by the quantum dynamics and its semiclassical approximation sought.

The general procedure given above is illustrated in three simple examples in the next three subsections.

C. Spin-j system

The Hilbert space is complex $2j+1$ dimensional or real (4j + 2) dimensional and the projective Hilbert space is CP^{2j}. As basic operators we choose three hermitian matrices S_i satisfying $[S_i, S_j] = i\hbar\epsilon_{ijk}S_k$. We also have the relation, $\langle\Psi|\sum \hat{S}_i^2|\Psi\rangle = j(j+1)\hbar^2$. Furthermore, $\langle\hat{S}_i^2\rangle = (x^i)^2 + \Delta S_i^2 \geq (x^i)^2$. Therefore, $r^2 := \sum(x^i)^2 \leq j(j+1)\hbar^2$.

The equivalence relation is defined in terms of $S^i(\Psi) := \Psi^\dagger S_i \Psi$ which gives a 3 dimensional Γ and $\mathcal{V}_S$ subspaces of tangent spaces of the CP^{2j}. Correspondingly, the symplectic structure gives $\omega(X_{S_i}, X_{S_j}) = \{S^i, S^j\}_q = \epsilon_{ijk}S^k$. Clearly $\mathcal{V}_S$ cannot be a symplectic subspace and indeed $Y := S^i(\Psi)X_{S_i}$ is also a vector field valued in $\mathcal{V}_S^\perp$. Denoting the coordinates on Γ by $x^i = S^i$, the projections are given by, $\xi_i := \pi_* X_{S_i} = \epsilon_{ijk}x^k\partial_j$. These projected vectors are *not* independent. Let $\vec{\beta}_I,\ I = 1, 2$ be any two, 3 dimensional vectors and define $\zeta_I := \beta_I^i\xi_i$. Their commutator is given by $[\vec{x}\cdot\vec{\beta}_J\ \beta_I^i\partial_i - \vec{x}\cdot\vec{\beta}_I\ \beta_I^i\partial_i]$. Clearly, if the two vectors $\vec{\beta}_I$ are orthogonal to the 'radial vector' $\vec{x}$, then the two vector fields commute. Let us further choose the two vectors $\vec{\beta}_I$ to be mutually orthogonal in anticipation. The two dimensional integral sub-manifolds are defined by $\phi(\vec{x}) = \phi(\sum x^i x^i) =$ constant, which are 2-spheres as expected for Γ_{cl}. Note that this is true for all spins.

Let us compute the induced symplectic structure. The 2-form α is defined by,

$$\alpha(\zeta_I, \zeta_J) := \beta^i_I \beta^j_J \tilde{\omega}(\xi_i, \xi_j) = \beta^i_I \beta^j_J \omega(X_{S_i}, X_{S_j}) = \beta^i_I \beta^j_J \epsilon_{ijk} S^k = \vec{r} \cdot \vec{\beta}_I \times \vec{\beta}_J . \tag{7}$$

Thus we have several spheres of radii r as candidate classical phases spaces and the normalization of the induced symplectic structure depends on both the particular sphere as well as arbitrary magnitudes of the two $\vec{\beta}_I$. The arbitrariness due to the magnitudes can be fixed by the choice of coordinates provided by the integral curves of the commuting vector fields. The normalization of β_I can be deduced by requiring $\zeta_1 := \partial_\theta,\ \zeta_2 := \partial_\phi$ where θ, ϕ are the usual spherical polar tangles. This requirement together with orthogonality of $\vec{r}, \vec{\beta}_I$, fixes the normalizations completely and leads to $\alpha(\partial_\theta, \partial_\phi) = r\ \sin(\theta)$.

For the case of $j = 1/2$, the Hilbert space is four (real) dimensional, the subset of normalized vectors, $\mathcal{S}$ is the three dimensional sphere and the state space $\mathcal{P}$ is the two dimensional sphere. The space Γ itself becomes *two* dimensional, the fibre over each point of Γ is zero dimensional. Thus there are no vertical vectors and the subspace $\mathcal{V}_S$ is *two* dimensional and symplectic. Thus $\Gamma_{cl} = \Gamma$ holds.

D. Particle on a circle

In this case the Hilbert space is the space of periodic (say) square integrable complex functions on the circle. We have three natural operators forming a closed commutator sub-algebra:

$$\begin{aligned}
\langle\phi|n\rangle &:= \frac{1}{\sqrt{2\pi}} e^{in\phi}, \qquad n \in \mathbb{Z} \qquad\qquad \text{basis functions} && (8)\\
\widehat{\cos}|n\rangle &:= \frac{1}{2}\left\{|n+1\rangle + |n-1\rangle\right\}; && (9)\\
\widehat{\sin}|n\rangle &:= \frac{1}{2i}\left\{|n+1\rangle - |n-1\rangle\right\}; &&\\
\widehat{\mathrm{p}}|n\rangle &:= \hbar n|n\rangle &&\\
\left[\widehat{\cos},\ \widehat{\sin}\right] &= 0 && (10)
\end{aligned}$$

$$\begin{aligned}
[\widehat{\cos},\ \widehat{\mathrm{p}}] &= -i\hbar\ \widehat{\sin} \\
\left[\widehat{\sin},\ \widehat{\mathrm{p}}\right] &= i\hbar\ \widehat{\cos} \\
\widehat{\cos}\ \widehat{\cos} + \widehat{\sin}\ \widehat{\sin} &= \mathbb{I}
\end{aligned} \tag{11}$$

Thus, for $\hat{F}_i$ we choose the $\widehat{\cos}$, $\widehat{\sin}$ and $\hat{p}$ operators and denote the corresponding quadratic functions $f_i(\Psi)$ by $\cos(\Psi)$, $\sin(\Psi)$ and $\mathrm{p}(\Psi)$. The Poisson brackets among these is obtained via $\{f_i, f_j\}_q = \omega(X_{f_i}, X_{f_j}) = \langle \frac{1}{i\hbar}[\hat{F}_i, \hat{F}_j]\rangle$. From the (11) it is easy to see that

$$\langle \cos^2\rangle + \langle \sin^2\rangle \ = \ \langle \cos\rangle^2 + \langle \cos\rangle^2 + (\Delta\cos)^2 + (\Delta\sin)^2 \ = \ 1. \tag{12}$$

The space Γ defined as the set of equivalence classes of rays having the same expectation values of the three basic operators, is the region in $\mathbb{R}^3 : (-1 < x < 1, -1 < y < 1, -\infty < z < \infty)$ where $x = \cos, y = \sin, z = \mathrm{p}$ and $0 \le x^2 + y^2 < 1$. Notice that the uncertainties in the trigonometric operators are also bounded from above.

It is easy to see that the span of the three Hamiltonian vector fields on the projective Hilbert space, X_{cos}, X_{sin}, X_p also contains a vector field, $Y = \cos X_{cos} + \sin X_{sin}$, which is valued in $\mathcal{V}_F^{\perp}$. From $\pi_* X_{f_i} = \{f_i, f_j\}_q \frac{\partial}{\partial x^j}$ it follows that the projection of these vectors fields to Γ are given by,

$$\pi_* X_{cos} \ = \ -y\frac{\partial}{\partial z} \quad , \quad \pi_* X_{sin} \ = \ x\frac{\partial}{\partial z} \quad , \quad \pi_* X_p \ = \ -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x} \quad , \quad \pi_* Y \ = \ 0\ . \tag{13}$$

Clearly all three projected vectors at any point of Γ are not independent and the 'radial' vector field $x\partial_x + y\partial_y$ on Γ is not a projection of any vector field valued in $\mathcal{V}_F + \mathcal{V}_F^{\perp}$. As an example, the two independent vectors $\zeta_1 := \pi_*(-\sin X_{cos} + \cos X_{sin})$ and $\zeta_2 := \pi_* X_p$ commute. The two dimensional vector spaces spanned by these, define a two dimensional integrable distributions on Γ. Introducing the usual polar coordinates in the $x - y$ plane, $x := r\cos\phi, y := r\sin\phi$, the two vector fields can be expressed as $\zeta_1 = r^2\partial_z,\ \zeta_2 = -\partial_\phi$. The integral sub-manifolds are defined by the integral curves of these vector fields and are characterized by $r = \text{constant}(\neq 0)$. Thus the classical phase space manifolds are cylinders as expected.

To obtain a symplectic structure α, on any of these cylinders, define $\alpha(\zeta_1, \zeta_2) := \tilde{\omega}(-\sin X_{cos} + \cos X_{sin}, X_p) = -\sin\{\cos, \mathrm{p}\} + \cos\{\sin, \mathrm{p}\} = \sin^2 + \cos^2 = r^2$. This is very much like the usual symplectic structure on a cylinder. Indeed, the polar coordinates introduced suggest that ϕ and $p_\phi := z/r^2$ be identified as the usual canonical variables on the cylinder.

For $r = 0$, of course one does not have a classical phase space (the cylinder degenerates to a line). Since we are interested in constructing classical phases spaces with a view to constructing a semiclassical approximation, we exclude the degenerate case. This also means that we exclude the fibres over the line ($x = 0 = y$). These fibres consists of all normalized states of the form $|\Psi\rangle = \sum_{n\in\mathbb{Z}} \psi_n|n\rangle$ with the coefficients satisfying the condition that if $\Psi_n \neq 0$ for some n then $\Psi_{n\pm1} = 0$. Analogous considerations will be more relevant in the next example.

In both the examples above, the Hilbert space has been separable and the classical phase spaces are obtained as embedded sub-manifolds of Γ. Both these features will change in the next example.

E. Isotropic loop quantum cosmology

This is an example with a non-separable Hilbert space [3, 4] but in many ways still similar to the particle on a circle example. We can use the so-called triad representation whose basis states are labelled by eigenvalues of the self-adjoint triad operator $\widehat{p}$ which take *all* real values with corresponding eigenstates being normalized. Unlike the particle on a circle example, the Hilbert space is not made up of *periodic* (or quasi-periodic) functions on a circle, but consists of *almost periodic* functions of a real variable c. The Hilbert space carries a *non-weakly-continuous* unitary representation of the Heisenberg group such that there is no operator $\widehat{c}$ generating translations of eigenvalues of $\widehat{p}$. However, exponentials of ic are well defined operators. In this regard, this is similar to the particle on a circle. Thus we can define:

$$\begin{aligned}
\langle c|\mu\rangle &:= \frac{1}{\sqrt{2\pi}}e^{i\mu c}, \qquad \mu\in\mathbb{R} \qquad\qquad \text{basis functions} && (14)\\
\widehat{\cos}_\lambda|\mu\rangle &:= \frac{1}{2}\left\{|\mu+\lambda\rangle + |\mu-\lambda\rangle\right\}; && (15)\\
\widehat{\sin}_\lambda|\mu\rangle &:= \frac{1}{2i}\left\{|\mu+\lambda\rangle - |\mu-\lambda\rangle\right\}; \\
\widehat{\mathrm{p}}|\mu\rangle &:= \hbar\mu|\mu\rangle \\
\left[\widehat{\cos}_\lambda,\ \widehat{\sin}_\lambda\right] &= 0 && (16)\\
\left[\widehat{\cos}_\lambda,\ \widehat{\mathrm{p}}\right] &= -i\hbar\ \lambda\ \ \widehat{\sin}_\lambda \\
\left[\widehat{\sin}_\lambda,\ \widehat{\mathrm{p}}\right] &= i\hbar\ \lambda\ \ \widehat{\cos}_\lambda
\end{aligned}$$

$$\widehat{\cos}_\lambda\ \widehat{\cos}_\lambda + \widehat{\sin}_\lambda\ \widehat{\sin}_\lambda \ = \ \mathbb{I} \tag{17}$$

The label λ indicates that we can define such 'trigonometric' operators for every *non-zero real* number λ. Since $\widehat{\cos}_{-\lambda} = \widehat{\cos}_\lambda$ and $\widehat{\sin}_{-\lambda} = -\widehat{\sin}_\lambda$, we will take $\lambda > 0$. We will denote by $\mathcal{L}_\lambda$, the Lie algebra defined by (16) for any fixed λ. In the circle example also we could define such operators, but the periodicity would restrict the λ to integers. The trigonometric operators for different λ's of course commute. The Hilbert space carries a *reducible* representation of the Lie algebra $\mathcal{L}_\lambda$ for every λ while every (countable) subspace $\mathcal{H}_{a,\lambda}$, spanned by vectors of the form $\{|a + k\lambda\rangle, k \in \mathbb{Z}\}$, gives an irreducible representation of $\mathcal{L}_\lambda$, for every $a \in [0, \lambda)$.

The presence of λ leads to $\zeta_1 = \lambda r^2 \partial_z, \zeta_2 = -\lambda\partial_\phi$. The integral curves are defined by constant r and $\dot{\phi} = -\lambda$. The integral curves of ζ_2 are clearly closed and we can choose the scale of ζ_2 so that the curve parameter itself is ϕ, i.e. redefine $\zeta_2 := -\lambda^{-1}\zeta_2$. We see that the classical phase space is the cylinder as in the previous example.

But the expected classical phase space is supposed to be $\mathbb{R}^2$ whose Bohr quantization leads to the non-separable quantum Hilbert space. How does one see the linear classical phase space?

Now, unlike the previous example where the trigonometric operators must induce shifts by integers to respect the periodicity, here we have more possibilities. We could consider enlarging the set of basic operators by including $\widehat{\cos_\lambda}, \widehat{\sin_\lambda}$ for n distinct values of λ. Thus our space of equivalence classes, Γ will be $(2n + 1)$ dimensional. One can see easily that there are now $(2n - 1)$ independent vectors in $\mathcal{V}_F$ which will project to zero vector on Γ and we will be left with exactly *two* independent vector fields, closed under commutator, on Γ. Explicitly, these vector fields can be chosen as: $\zeta_1 = \pi_*(\sum_i \lambda_i\{-\sin_{\lambda_i} X_{cos_{\lambda_i}} + \cos_{\lambda_i} X_{sin_{\lambda_i}}\}) = \sum_i \lambda_i\{(x^i)^2 + (y^i)^2\}\partial_z$ and $\zeta_2 = \pi_* X_p = \sum_i\{-\lambda_i(x^i\partial_{y^i} - y^i\partial_{x^i})\}$. There is freedom available in defining ζ_1, but ζ_2 has freedom of *only* overall scaling. In the (x^i, y^i) planes, we can introduce the polar variables (r_i, ϕ_i) and see that the integral curves of ζ_1 are along the z-direction while those of ζ_2 are defined by the equations: $r_i = \text{constant}_i(\neq 0), \dot{\phi}^i = -\lambda_i$ i.e. winding curves on an n-torus, T^n. Evidently, for λ_i with irrational ratios, these integral curves are non-periodic. Thus we get a map of $\mathbb{R}$ into the T^n which however cannot be an embedding since the induced topology on the winding curve is not the standard topology on $\mathbb{R}$. One has only an immersion. Combining with the curve parameter of ζ_1, one has an immersion of $\mathbb{R}^2$ and this is adequate to define the symplectic form α. In effect we obtain the classical phase space which is topologically $\mathbb{R}^2$ and is immersed in an $\mathbb{R} \times T^n \subset \Gamma$.

The symplectic form is computed as before and leads to, $\alpha(\zeta_1, \zeta_2) = \sum_i \lambda_i^2 r_i^2$. Denoting the curve parameter for the vector field ζ_2 by Q and defining its canonically conjugate variable as $P := z/(\sum_i \lambda_i^2 r_i^2)$, one has the usual symplectic form on $\mathbb{R}^2$.

If *all* the λ_i happen to be rational numbers (or rational multiples of a single irrational number), then the integral curves of ζ_2 are closed and one would obtain the classical phase space to be the cylinder (or a covering space thereof). Clearly, to obtain the planar phase space, one must have at least *two* λ' whose ratio is an irrational number *with the corresponding* $r_i \neq 0$.

Here we find an example where depending upon the choice of basic operators or more precisely the parameter(s) λ_i, we can obtain *two different topologies* for the classical phase space. We also see that although the choice of basic operators can vary widely, one always obtains a *two dimensional* classical phase space which is generically $\mathbb{R}^2$.

All these mathematical procedures of constructing 'classical' phase spaces from the quantum one become physically relevant for studying semiclassical approximation(s) only when *a dynamics* is stipulated and attention is paid to the uncertaintities in the basic operators $\hat{F}_i$. These ultimately decide which choice of the Lie algebra of $\hat{F}_i$ is useful. We address these in the next section.

II. DYNAMICS AND SEMICLASSICALITY

In this section, we discuss how a quantum evolution taking place in the quantum phase space can generate a Hamiltonian evolution on a classical phase space constructed in the previous section. This will naturally involve a choice of a 'section'. While, every section will generate *a* classical dynamics, further conditions have to be imposed for the classical dynamics to be a good approximation to the quantum dynamics. These are the conditions of 'semiclassicality' and 'tangentiality'. The fact that the classical phase of the isotropic cosmology arises as an immersed manifold in the base space, introduces further considerations which are discussed next.

So far we just constructed a 'classical' phase space, Γ_{cl}, selecting a sub-algebra of self-adjoint operators and using the naturally available symplectic geometry of the quantum state space. This is completely independent of any dynamics. Consider now a quantum dynamics specified by a self-adjoint Hamiltonian operator, $\hat{H}$. The bundle structure allows one to project any quantum trajectory generated by X_H, to a trajectory in the base space Γ.

In the general case where a Γ_{cl} is a submanifold of Γ, the first problem is that projected trajectories may not even be confined to Γ_{cl} i.e. $\pi_*(X_H) \notin \pi_*(\mathcal{V}_F)$. Notice that X_H is not 'constant' along the fibres ($\mathcal{L}_v X_H$ is not vertical) and therefore we need to specify the points on the fibres. If there are *no* quantum trajectories whose projections remain confined to some Γ_{cl}, then the choice of the Lie algebra of $\hat{F}_i$ used in the quotienting procedure, *in conjunction with the Hamiltonian*, is inappropriate for developing a semiclassical approximation and one has to look for a different choice. Let us assume that one has gone through this stage and found a suitable Lie algebra so that there are at least some quantum trajectories whose projections are confined to Γ_{cl}. Note that when $\Gamma_{cl} = \Gamma$, as in the usual case of Heisenberg Lie algebra, this issue does not arise at all.

The second issue is whether there are 'sufficiently many' quantum trajectories whose projections are confined to Γ_{cl}. Sufficiently many would mean that projection of these trajectories will be at least an open set of Γ_{cl}. This in turn allows the possibility that at least a portion of the classical phase space can be used for a semiclassical approximation. Let us also assume this.

The third issue is whether the projected trajectories are generated by a Hamiltonian function on Γ_{cl} and whether such a Hamiltonian function can be constructed from the quantum Hamiltonian function $H(\Psi) = \langle\Psi|\hat{H}|\Psi\rangle$ already available on the quantum state space. This function however is not constant along the fibres and cannot be 'projected' down to the base space. One has to choose a section: $s : \Gamma \to \mathcal{P}$, $\pi \circ s = id$. There are infinitely many choices possible. However, for every such choice, we can define a function $\tilde{H} : \Gamma \to \mathbb{R}$ as the pull back of $\langle\Psi|\hat{H}|\Psi\rangle|_{s(\Gamma)}$. The restrictions of these functions to any of the sub-manifolds, Γ_{cl} would be candidate *effective Hamiltonians*, H_{eff}.

In this regard, we would like to note a simple fact. Let $s : \Gamma \to \mathcal{P}$ be a section (possibly local) and let X_H be a Hamiltonian vector field on $\mathcal{P}, H(\Psi) := \langle\Psi|\hat{H}|\Psi\rangle$. Let $\xi := \pi_*(X_H|_{s(\Gamma)})$, $\tilde{H}(x^i) := s^*(H|_{s(\Gamma)})$. The (symplectic) 2-forms are related as $\omega := \pi^*\tilde{\omega} \leftrightarrow \tilde{\omega} = s^*\omega$. Then it is true that ξ is a Hamiltonian vector field (on Γ) with respect to $\tilde{\omega}$, with $\tilde{H}$ as the corresponding function. This follows as,

$$\tilde{\omega}(\xi, \xi_i) \ = \ \omega(X_H, X_{f_i}) \ = \ X_{f_i}(H) \ = \ \pi_*(X_{f_i})(\tilde{H}) \ = \ \xi_i(\tilde{H}) \ \ \forall \ i. \tag{18}$$

Since any Γ_{cl} is a sub-manifold of Γ we have natural projection (restriction) and inclusion maps so that we obtain the restriction of ξ to tangent spaces of Γ_{cl} as the Hamiltonian vector field with $i^*\tilde{H}$ as the corresponding function. The net result is that for any section and any Hamiltonian vector field, we can always obtain *a* classical Hamiltonian description. Since there are infinitely many sections, we have infinitely many classical dynamics induced from a given quantum dynamics. Is any of these dynamics a 'good' approximation to the quantum dynamics? For this we have to invoke further conditions on the sections.

The expectation value functions are naturally observables and our restriction to expectation values of a subset of the observables $\hat{F}_i$, amounts to saying that in a given situation and with a given experimental capability, we can discern the quantum dynamics only in terms of these few 'classical' variables. With better experimental access, we may need more such variables. Generically, these observables also have quantum uncertainties and provided these uncertainties are smaller than the observational precision, we can ignore them, thereby justifying a classical description. Obviously, such a property will not be exhibited by all quantum states and not for all times. Consequently, one first defines a quantum state to be semi-classical with respect to a set of observables $\hat{F}_i$ provided the uncertainties $(\Delta F_i)_\Psi$, are smaller than some prescribed tolerances δ_i. For such a notions to be observationally relevant/useful, semi-classical states so defined must evolve, under quantum dynamics, into semi-classical states for *sufficiently long durations*. Since we identified a classical phase space using equivalence relation specified by a set of functions $f_i(\Psi) := \langle\Psi|\hat{F}_i|\Psi\rangle$, closed under the $\{\cdot, \cdot\}_q$, these are naturally the candidate observables with respect to which we can define semiclassicality of a state.

On each fibre then, we could find subsets for which the uncertainties would be smaller than some specified tolerances δ_i [2]. Quantum states within these subsets would be semi-classical states. Let us assume that we identify such bands of semiclassicality on each fibre. Our sections should intersect each fibre within its semi-classical band(s). This still leaves

[2] The tolerances could be prescribed to vary over different regions of Γ. For instance, for larger expectation values, one could have less precision and thus permit larger values of δ_i.

an infinity of choices and is still not enough to have the classical trajectories generated by an effective Hamiltonian to approximate the projections of quantum trajectories.

That the projection of a Hamiltonian vector field restricted to $s(\Gamma)$ is also a Hamiltonian vector field on Γ_{cl} follows independent of whether or not the Hamiltonian vector field X_H is *tangential* to the section. However, tangentiality of the Hamiltonian vector field X_H to a section is necessary so that projection of a quantum trajectory in $\mathcal{P}$, generated by X_H, gives the corresponding classical trajectory (in Γ) generated by $\xi_{\tilde{H}}$. If tangentiality fails, then the pre-image of a classical trajectory, will *not* be a single quantum trajectory i.e. pre-images of nearby tangent vectors of a classical trajectory will belong to *different* quantum trajectories. Once X_H is tangential to $s(\Gamma)$, the classical evolution in Γ will mirror the quantum evolution (and by assumption made at the beginning, classical evolution will be confined to Γ_{cl}).

Thus, to develop a semi-classical approximation what is needed is to guess or identify a classical phase space Γ_{cl} such that sufficiently many quantum trajectories project into Γ_{cl}, select criteria of semiclassicality and *choose a suitable (and possibly local) section which is tangential to the X_H and lies within a semi-classical band.* The classical phase space is obtained via the quotienting procedure while the effective classical Hamiltonian is obtained as the pull-back of $\langle \hat{H} \rangle$. Note that we have *not* assumed that the Hamiltonian operator is an algebraic function of the basic operators chosen in the quotienting procedure. The effective Hamiltonian however is always a function on Γ_{cl} (and $\tilde{H}$ is a function on Γ) by construction. Even if the Hamiltonian operator is an algebraic function of the basic operator, the $\tilde{H}$ will *not* be an algebraic function of the expectation values of basic operators.

Suppose now that we have chosen a section (over Γ) satisfying all the conditions above. In the general case where $\Gamma_{cl} \neq \Gamma$, we have to restrict to the section over a Γ_{cl} i.e. restrict to those states in the section which project into Γ_{cl}. Secondly, in general we will not have operators corresponding to the canonical coordinates in Γ_{cl} and hence for the semiclassical criteria we have to use only the basic operators used in obtaining Γ. The LQC example, reveals implications of these aspects. We will see that the Hamiltonian operator also needs to have non-trivial dependences on all the basic operators chosen for the quotienting procedure and the semiclassicality criterion also needs to be phrased differently.

Recall that to construct the classical phase space $\mathbb{R}^2$ (as opposed to a cylinder), one needs to use at least two sets of trigonometric operators with labels λ, λ' such that (i) λ'/λ is irrational and (ii) r, r' both being non-zero. If a state in a section has support only on a lattice generated by λ (say), or a subset thereof, one can see immediately that $\cos_{\lambda'} = 0 = \sin_{\lambda'}$ and hence $r' = 0$. Therefore such states in a section will not represent points in the classical phase space (but will represent points in a cylinder). Let us then assume that our section consists of states having support on (sub-)lattices generated by both λ, λ'. To be definite, let us take, $|\Psi_{a,\lambda}\rangle$ to be a normalized vector which is a linear combination of vectors of the form $|a + k\lambda\rangle, k \in \mathbb{Z}, a \in (0, \lambda)$ and likewise $|\Psi_{b,\lambda'}\rangle$. Let a state in a section be $|\Psi\rangle := (|\Psi_{a,\lambda}\rangle + |\Psi_{b,\lambda'}\rangle)/\sqrt{2}$. We are now guaranteed to have the expectation values of $\hat{p}, \widehat{\cos}_\lambda, \widehat{\sin}_\lambda, \widehat{\cos}_{\lambda'}, \widehat{\sin}_{\lambda'}$ to determine a unique point of the classical phase space, $\mathbb{R}^2$. Furthermore, for any basic trigonometric operator with a label λ'', with λ'' irrational multiple of λ, λ', its expectation value in the state $|\Psi\rangle$ will be zero and the uncertainty will be 1/2. For the uncertainty, we note the identities, $\widehat{\cos^2}_{\lambda''} = (1 + \widehat{\cos}_{2\lambda''})/2$ and $\widehat{\sin^2}_{\lambda''} = (1 - \widehat{\cos}_{2\lambda''})/2$. We would like to see evolution of these quantities generated by a self-adjoint Hamiltonian operator, $\hat{H}_{\lambda_0}$ which is a function only of $\hat{p}, \widehat{\cos}_{\lambda_0}, \widehat{\sin}_{\lambda_0}$ for some fixed λ_0. The quantum evolution of $\langle\cos_{\lambda''}\rangle, \langle\sin_{\lambda''}\rangle$ will be given by the expectation values of the commutators of the corresponding trigonometric operator with the Hamiltonian operator,

The Hamiltonian acting on $|\Psi\rangle$ will generate vectors with labels $k\lambda + l\lambda_0$ and $k\lambda' + l\lambda_0$. The trigonometric operator acting on $\langle\Psi|$ on the other hand will generate vectors with labels $k\lambda \pm \lambda''$ and $k\lambda' \pm \lambda''$. If $\lambda_0, \lambda, \lambda', \lambda''$ are all incommensurate, then the inner product of these states will be zero and hence the expectation value of the commutator will be zero. Consequently, expectation values of these trigonometric operators will not evolve. By the same logic, their uncertainties also will not evolve. If the Hamiltonian operator is a sum of a function of $\hat{p}$ alone plus a function of the trigonometric operator alone (as could happen for polymer quantization of usual systems [4]), then of course expectation values and uncertainties of trigonometric operators with both labels λ, λ', will evolve, irrespective of λ_0. Barring this exception, the only way to get a non-trivial evolution is to have λ_0 to equal λ or λ'. Suppose we choose $\lambda = \lambda_0$ (The quantum Hamiltonian is given and we can choose the states in the section so as to develop a semiclassical approximation). Now, the evolution of $\langle\widehat{\cos_{\lambda_0}}\rangle$ and $\langle\widehat{\sin_{\lambda_0}}\rangle$ will be non-trivial and so also the evolution of the corresponding uncertaintities. However, the evolution of corresponding quantities for λ' will still be trivial! In effect, the projection of the quantum trajectories on the classical phase space will *not* be non-periodic curves.

Thus, to have a non-trivial classical evolution (in Γ_{cl}), one will have to have (a) trigonometric operators with an incommensurate set of λ's (at least two), (b) a section satisfying semiclassicality and tangentiality, (c) states (in a section) involving (sub-)lattices corresponding to these λ's and (d) the quantum Hamiltonian also involving trigonometric operators with the chosen set of λ's. Note that this is a statement about developing a semiclassical approximation and does *not* imply or indicate any inconsistency of the choice of the Hamiltonian operator at the fundamental quantum level.

Having multiple incommensurate λ's also affects the uncertaintities in the trigonometric operators. Continuing with the choice of just λ, λ', one sees that,

$$(\Delta \cos_\lambda)^2_\Psi = \frac{1}{4} + \frac{1}{4}\langle \Psi_{a,\lambda}|\widehat{\cos}_\lambda|\Psi_{a,\lambda}\rangle^2 + \frac{1}{2}(\Delta \cos_\lambda)^2_{\Psi_{a,\lambda}} \tag{19}$$

The first term comes from $|\Psi_{b,\lambda'}\rangle$ piece of the wave function. Similar expressions are obtained for the other three other trigonometric operators. Notice that these uncertainties are always larger than 1/4. If we choose a state based on N incommensurate λ's, then the first term changes to $(N-1)/(2N)$, the coefficient of the second term changes to $(N-1)/N^2$ while the coefficient of the third term changes to $1/N$. For large N, the uncertaintities become 1/2. This is irrespective of the details of the states. Thus, if the semiclassicality criteria required uncertainties in the trigonometric operators to be small (recall that uncertainties are bounded above by 1 (12)), then *no* state involving several incommensurate λ's will satisfy the criterion of semiclassicality. Since for $N = 1$ (single λ), only the last term survives, a way out is to require the uncertainties in $\widehat{\cos}_\lambda, \widehat{\sin}_\lambda$ *in the states of the form* $|\Psi_{a,\lambda}\rangle$ to be smaller than prescribed tolerances. This extra feature arises because the classical phase space Γ_{cl}, is immersed in a complicated way in the base space Γ and one does not have operators corresponding to the canonical coordinates on Γ_{cl} which could be used in formulating semiclassicality criteria.

III. DISCUSSION

In this work, we have explored a particular strategy of developing a semiclassical approximation, namely, systematically, constructing a classical Hamiltonian system using the available quantum structures such that at least some quantum motions can be faithfully viewed as classical motions within certain precisions. This was done by exploiting the symplectic structure of the quantum state space which is typically infinite dimensional. One first constructs a (finite) dimensional classical phase space by a quotienting procedure which views the quantum state space as a bundle over a base space Γ. There are several such phase spaces one can construct depending on the choice of a Lie algebra of basic operators. Generically, one gets classical phase space, Γ_{cl} as a submanifold of the base space. So far, typically the special case wherein $\Gamma_{cl} = \Gamma$ has been analysed in the literature, eg [1, 2, 4, 5]. We have given a generalization of the procedure. The main difference that occurs in the general case is that the vertical subspace and the space spanned by the Hamiltonian vector fields corresponding to the basic operators, have a non-trivial intersection and also together they do not span the tangent space of the bundle. We illustrated the general procedure for three different types of quantum systems - none resulting from a weakly continuous representation of the Heisenberg group. Note that if the view of semiclassical approximation mentioned above is to have a general enough validity, then it is necessary to be able to construct non-linear phase spaces as well and the quotienting procedure given here shows that it is possible.

A useful semiclassical approximation however, cannot be developed without reference to dynamics. Construction of effective classical dynamics induced from the quantum one required the choice of a section of the bundle. Once a section (*any section*) is chosen, one can immediately define an effective Hamiltonian (and other effective functions) on the classical phase space and a corresponding classical dynamics. However, this associated classical dynamics will be a poor approximation to the underlying quantum mechanics *unless* the sections are further restricted. We discussed the necessary conditions on the section. The two main conditions are those of semiclassicality and tangentiality. However, when $\Gamma_{cl} \neq \Gamma$, one needs additional condition namely, there should be sufficiently many quantum motions which will project to curves in Γ_{cl} (and not just in Γ) and that these comprise a (local) section.

The polymer state space brought out further features. To have a non-trivial motion on Γ_{cl}, one needs to enlarge the set of basic operators to include trigonometric operators with a set of incommensurate λ's, use states which are based on (sub-)lattices generated by these λ's *and* have the Hamiltonian operator also depend on trigonometric operators with many λ's, except when the Hamiltonian operator has a additively separated dependence on the trigonometric operators and the operator $\hat{p}$. Furthermore, the semiclassicality criteria also needs to be applied to the trigonometric operators such that the uncertaintities for the operators labelled by λ are computed with states based on (sub-) lattice generated by the same label λ. In effect, one chooses states based on the various $\lambda-$ lattices and chooses a linear combination of these states to form a (local) section. How to choose the set of incommensurate λ's is not clear at this stage. A detailed construction of a semiclassical approximation for LQC, following the steps discussed here, needs to be done. The new Hamiltonian operator proposed in [6] looks promising since it is self-adjoint and also naturally connects triad labels in a specific irregular lattice. A detailed analysis of the original LQC Hamiltonian of [3], with coherent states based on a single $\lambda-$lattice is available in [5].

There is an alternate way to develop a semiclassical approximation [7]. This also uses the geometric view of quantum mechanics and looks at the Hamiltonian flow on the quantum phase space directly. For the special case of Hilbert space carrying the weakly continuous representation of the Heisenberg group, one can introduce the so-called *Hamburger momenta* variables to introduce suitable (adapted to the bundle structure) coordinates on the quantum state space. The exact quantum dynamical equations can then be viewed as the Hamilton's equations of motion. Depending upon the quantum Hamiltonian function, evolution of most of these variables could decouple from those of a smaller (finite) set of variables. This smaller set of variables would then constitute a classical approximation i.e. be thought of as classical degrees of freedom. In effect, the evolution of the remaining 'quantum degrees of freedom', control how the uncertaintities in the classical degrees of freedom evolve. Violation of semi-classicality can then be viewed as coupling of the evolutions of the classical and the quantum degrees of freedom. This method would be more useful to track when the quantum evolution can exit the semiclassical bands signalling break down of semiclassical approximation. So far this method has been available in the context of Schrodinger quantization. Such an explicit description of quantum evolution probably has to be done on a case by case basis.

Clearly, within the restrictions provided by semiclassicality and tangentiality, one can imagine different approximation schemes which will systematically construct a better and better semi-classical approximation. A natural way to phrase such a procedure is to formulate it in terms of a family of sections. Thus one may begin with a section as giving leading classical approximation and systematically change it to improve the approximation. The usual perturbative approach can be viewed as beginning with a section defined by "free particle states" and adding corrections to it get the new section closer to tangentiality. This would be an interpretation of inclusion (or computation) of quantum corrections.

One also encounters a situation wherein new degrees of freedom are excited beyond a certain threshold scale. This will mean that merely changing sections will *not* ensure tangentiality and semiclassicality. In the language of [7], some of the 'quantum degrees of freedom' have to be thought of as new 'classical degrees of freedom'. One has to include further 'basic' operators and use a new quotienting procedure. Now one can repeat the analysis and it is certainly conceivable that the true dynamics does satisfy the two requirements with these additional variables defining sections.

Thus the geometrical view point shows that from a quantum perspective, inadequacy of a classical approximation can arise in two ways – inappropriate choice of section which could be improved perturbatively in some cases and/or inadequate choice of basic variables used in the quotienting procedure.

The present analysis leaves out two important classes of systems: finite dimensional constrained systems and field theories with or without constraints. We hope to return to these in the future.

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